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ON THE RELATION BETWEEN INDISTINGUISHABILITY OF IDENTICAL PARTICLES AND (ANTI)SYMMETRY OF THE WAVE FUNCTION IN QUANTUM MECHANICS

ABSTRACT. Two different concepts of distinguishability are often mixed up in attempts to derive in quantum mechanics the (anti)symmetry of the wave function from indistinguishability of identical particles. Some of these attempts are analyzed and shown to be defective. It is argued that, although identical particles should be considered as observationally indistinguishable in (anti)symmetric states, they may be considered to be conceptually distinguishable. These two notions of (in)distinguishability have quite different physical origins, the former one being related to observations while the latter has to do with the preparation of the system.

1. INTRODUCTION

In the elementary wave function formulation of quantum mechanics a system of identical particles (SIP) is required to have a wave function that is either totally symmetric or totally anti-symmetric under a permutation of the particle variables,

(1) $$\Psi(P(x_1, \ldots, x_n)) = \pm\, \Psi(x_1, \ldots, x_n).$$

This (anti)symmetry (AS) requirement is often introduced as a postulate (Dirac, 1958; Schiff, 1955; Messiah, 1958) but it is sometimes thought to be derivable from the indistinguishability (ID) of the identical particles (Landau and Lifshitz, 1959; Blokhintsev, 1964). The reasoning corresponding to this line of thought, is as follows: since the particles are indistinguishable, a permutation of the particles cannot have observable consequences. Therefore, the expectation values of any observable $f(x_1, \ldots, x_n)$ should be the same in states $\Psi(x_1, \ldots, x_n)$ and $\Psi(P(x_1, \ldots, x_n))$. From this it is concluded that the equality

(2) $$\Psi(P(x_1, \ldots, x_n)) = c\Psi(x_1, \ldots, x_n),\ |c| = 1,$$

should be valid. This implies one-dimensionality of the representations of the permutation group. Since only the symmetric and the anti-symmetric representations of the permutation group are one-dimensional, ID entails AS, if this derivation is applicable.

From several directions objections have been raised (Messiah and Greenberg, 1964; Hestenes, 1970; de Muynck, 1975) against this

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derivation of AS from ID. Thus, by Messiah and Greenberg (1964) it is shown that AS is stronger than ID. So, additional assumptions, besides ID, must be used in a derivation of AS. Hestenes (1970), in discussing the Gibbs paradox, stresses the conceptual difference of ID and AS. Finally, by de Muynck (1975) both these arguments are combined, thus exhibiting the nature of the additional assumptions mentioned above.

Since that time there have been several discussions (Kaplan, 1976; Sarry, 1979; Shadmi, 1978) of the relation between AS and ID, two of these discussions (Kaplan, 1976; Sarry, 1979) being renewed attempts to derive AS from ID. My purpose here is to analyse these derivations and demonstrate their defectiveness. This will be done in sections 7–9. Since this analysis is dependent on our understanding of the notion of (in)distinguishability, we will first give a discussion of this concept.

2. TWO KINDS OF DISTINGUISHABILITY

It is necessary, in order to arrive at a better understanding of ID, to draw a distinction between two different concepts of distinguishability, viz., *conceptual* distinguishability (CD) and *observational* distinguishability (OD). We will refer to two physical entities as conceptually distinguishable whenever it is possible to provide each with a label or name by which it can be represented in the theoretical description. We will refer to two physical entities as observationally distinguishable if a physical exchange of the entities leads to a state that is observationally different, that is, if we have at our disposal a measurement procedure of at least one observable having different expectation values in the two states.

From an empiricist's point of view it might seem that only the second mode of distinguishability makes sense, because the label would not have any observable meaning if a physical exchange resulted in a state that is observationally the same. However, as is well known since the days of Hume, physics transcends empiricism in attributing the property of *substance* to the set of phenomena that can be interpreted as the occurrence of a particle. This idea of substance is the expression of our belief that the particle has some permanence, and therefore a certain individuality which is kept throughout the observed process. It is this idea of substance that makes it possible to interpret the solution of Newton's equation as describing the motion of one and the same particle, to be marked by one single label during the whole process.

Conceptual distinguishability of particles is just an extension of this idea to systems of more than one particle. By Shadmi (1978) indistinguishability of particles is brought into relation with Leibniz's principle of the "identity of the indiscernibles". This seems to make sense if "indiscernible" is equated to "conceptually indistinguishable": if it is impossible to distinguish two entities by means of any label, they should be identified. However, as was noted already by Margenau (1944), Leibniz's principle has no bearing on the notion of observational indistinguishability (OID): even if we do not label the two particles, it is easily demonstrated that a system of two particles is observationally different from a one-particle system; this can be done by simply counting the particles. Therefore, OID is no reason for identification of the particles.

CD is not only a presupposition of classical mechanics, but also of quantum mechanics. In the n-particle wave function $\Psi(x_1, \ldots, x_i, \ldots, x_n)$ the symbol x_i represents the position of the ith particle. Clearly, it is a presupposition of the wave function formalism that each particle is labeled by means of an *index*. This kind of labeling, moreover, is presupposed both for nonidentical and for identical particles.

OD without CD would have the paradoxical consequence that two states differing only by an exchange of two particles could only be described by the same theoretical entity, and yet would be observationally different. For this reason CD is necessary for OD: the label should be there in order to be observable. However, CD is not sufficient. In order to be observationally distinguishable the particles should have labels that actually *are* observable. As was noted by Hestenes (1970), the meaning of observational (in)distinguishability is determined by the nature of the experiments which can be performed on the system. If there do not exist experimental means to observe the particle labels, the particles are observationally indistinguishable, even if they are conceptually distinguishable.

3. ID IN CLASSICAL AND IN QUANTUM MECHANICS

It is often asserted (e.g., Jauch, 1966) that indistinguishability of identical particles is a typically quantum mechanical property, classical particles being distinguishable by their trajectories. Since quantum mechanical particles do not have trajectories, this might even be used as an argument against the possibility of labeling particles in quantum mechanics, and therefore against *conceptual* distinguishability of identical particles. If the positions of all *n* identical particles are measured on two consecutive moments, quantum mechanics does not offer any warrant that a physical exchange of particles between the two measurements has not taken place. For identical particles this would look like an exchange of labels between the particles. This would entail the unsuitability of a label as an expression of a permanent individuality of a particle.

As was stated before, classical as well as quantum mechanics start from the presupposition that a particle can be labeled by means of a particle index, also in case the particles are identical. In the case of nonidentical particles this does not pose any conceptual problems in quantum mechanics. In our opinion, the situation is quite analogous in the case of identical particles. It would seem rather strange if a particle would lose its individuality as soon as a particle of the same type would enter its universe. For this reason, from a conceptual point of view the situation does not seem different for identical particles. Although in quantum mechanics the idea of a particle trajectory is lost in a strict sense, Ehrenfest's theorem nevertheless can provide us with the notion of particle trajectories even if the wave packets of the particles are overlapping. Therefore, there does not seem to exist any a priori objection against describing, also in quantum mechanics, identical particles as *indexed* particles, even if they are thought to be observationally indistinguishable. As to *conceptual* distinguishability, classical and quantum mechanics can be treated on a par.

According to Hestenes (1970), there is also no difference between classical and quantum mechanics with respect to *observational* distinguishability. OD hinges on the experimental possibility of observing the particle label. We completely agree with Hestenes that this is a question that (a) can be decided only in an operational way, and (b) is independent of whether the context is classical or quantum mechanics. As a matter of fact, if we consider a volume of a gas of identical particles (as in the context of the Gibbs paradox), then *also classically* we have no experimental means of distinguishing the particles in the gas. On the other hand, if the particles are far apart, the particles, both classically and quantum mechanically, can be distinguished by marking the region each particle is in (Schiff, 1955; Mirman, 1973).

It would seem that in the quantum mechanical case the above-mentioned way to distinguish the particles would be frustrated by the (anti)symmetry of the wave function. However, the question is whether this objection provides an argument against distinguishability, or rather against the application of AS under the given circumstances. We will see in section 5 that certain quantum physical SIP's are described more accurately by a non-AS state than by an AS state.

In our opinion, Hestenes correctly views AS and ID as physically unrelated. AS has no other meaning than the equality

(3) $$|\Psi(P(x_1, \ldots, x_n))|^2 = |\Psi(x_1, \ldots, x_n)|^2,$$

stating that the probability of finding the particles $1, \ldots, n$ at positions $x_1, \ldots, x_n$ equals the probability of finding the particles in a configuration in which the particles have been permuted. This equality has nothing to do with "a physical exchange of particles", which sometimes is attributed in a more or less pictorial way to an "exchange interaction", intended to explain the appearance of "exchange terms" in the energy eigenvalues. Such a physical exchange would contradict the idea of inertia underlying conceptual distinguishability, which is always presupposed here. AS, as expressed by (3), reveals a physical equivalence of the particles, which should be viewed in our opinion, as a consequence of the way the (anti)symmetric state $\Psi(x_1, \ldots, x_n)$ *has been prepared* (de Muynck, 1975). Far from being a consequence of ID it seems that the equality (3) can be put to a direct experimental test only if the particles are observationally distinguishable. For instance, consider an electron scattering experiment, the electrons stemming

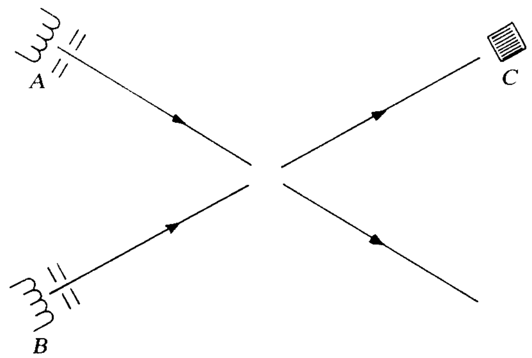

Fig. 1. Electron scattering experiment.

from two electron guns, A and B (cf. figure 1). If we give the electron, generated by $A(B)$, the index 1(2), then observational distinguishability is necessary in order to determine whether electron 1 or electron 2 is detected by detector C.

4. DESCRIPTIONS OF SIP's IN CLASSICAL AND QUANTUM MECHANICS

When identical particles are considered to be indistinguishable in literature, this is generally intended to refer to *observational* indistinguishability. Observations being described in quantum mechanics by quantum mechanical observables (self-adjoint operators), observational (in)distinguishability of identical particles amounts to the question of which quantum mechanical observables are physically relevant, that is, for which observables a measurement procedure can be conceived. For instance, the position observable x_i of the ith particle can have only physical relevance if our measuring apparatus can somehow recognize the particle index. Since for identical particles this does not seem to have a direct operational meaning, observables like x_i are often denied physical relevance. Only those observables which are invariant under a permutation of the particle indices (*symmetrical* observables) are then thought to have physical relevance.

Strictly speaking, this whole reasoning should be applied also in classical mechanics, at least in those situations where it is *practically* impossible to distinguish the particles observationally (as for instance in a classical statistical mechanical description of a volume of gas). This has been done already by Gibbs (1902) who replaced a description in terms of so-called *specific* phases (Γ-space: $(q_1, p_1, q_2, p_2, \ldots, q_n, p_n)$) by one using *generic* phases (μ-space: (q, p)), the latter one ignoring particle indices. The two descriptions give the same results for the macroscopic observables of statistical mechanics if states, differing in the Γ-space description only by a permutation of the particles, are identified, and associated with one and the same state on μ-space.

An analogous situation obtains in quantum mechanics. Here, second quantization (quantum field theory) provides us with a description of identical particles which, for symmetrical observables, is equivalent to the elementary wave function description (Robertson, 1973; Greenberg and Raboy, 1982). As in the μ-space description, in second quantization the particles do not have indices. This makes these theories extremely suitable for the description of systems of observationally indistinguishable particles. However, this does not imply, conversely, that particles described by second quantization theories are necessarily observationally indistinguishable. Second quantization can describe a system of particles that are far apart. When two identical particles cross a bubble chamber simultaneously without colliding, they produce two completely separated tracks, by which the particles can be distinguished observationally. Admittedly, this method of distinguishing the particles observationally is less direct than e.g., distinguishing a red and a white billiard ball by means of their color. It does not rule out the theoretical possibility that the particles exchange their places while following their tracks, without in any way disturbing these tracks. This possibility, however, would not only impair observational distinguishability but also conceptual distinguishability. Although, from a strictly empiricist point of view, this can hardly be appreciated as a counterargument, we are prepared to take it seriously for reasons discussed in the preceding sections. As a matter of fact, denying the *possibility* of conceptual distinguishability in quantum field theories would signify a fundamental break with the mechanistic world view that manifests itself in the wave function formulation of quantum mechanics through the use of particle indices.

As the bubble chamber experiment shows, it is possible to obtain experimental information on the properties of the *individual* particles, even if these particles are parts of an SIP. Such *information* can *not* be described by the second quantization formalism since this formalism contains only counterparts for those observables that are symmetric under permutation of the particle indices (Robertson, 1973; Greenberg and Raboy, 1982). Thus, there exists a field operator corresponding to the "total position operator" (Mathews and Esrick, 1980) $\sum_{i=1}^{n} x_i$. However, in the unindexed formalism we have no field operator corresponding to x_i. Such operators simply have no meaning in quantum field theory. Second quantization describes only those observations in which the particle indices are ignored. Distinguishing identical particles observationally requires labeling of the particles. For this reason OD can not be dealt with by the second quantization formalism. One could say that as far as this formalism is concerned, the particles are observationally indistinguishable. Because of the equivalence with the wave function formalism, second quantization is not incompatible with CD. Contrary to the latter formalism, CD is manifest in the former. It is precisely for this reason that the wave function formalism offers the possibility to handle OD. In this respect it is a more powerful formalism than second quantization.

5. AS AND OID

It should be noticed here that also the indexed theory would not be of much help in distinguishing the particles observationally if the states of the system would be restricted to states obeying AS, since in that case

(4) $\langle x_1\rangle_{AS} = \cdots = \langle x_i\rangle_{AS} = \cdots = \langle x_n\rangle_{AS}$,

or, more generally

(5) $\langle A\rangle_{AS} = \langle P^{\dagger}AP\rangle_{AS}$

for *any* operator A. This trivially follows from (2). For our electron scattering experiment (figure 1) the equality (4) expresses the equal probabilities for particles 1 and 2 to be in either of the two outgoing beams. From (4) and (5) we see that in AS states the particles are observationally indistinguishable. Evidently in such states the particles are mixed (Hestenes, 1970) so thoroughly that, as in the mixing of two volumes of a gas, the individual particles become completely equivalent, and each particle has equal probability to be found in those places which are allowed by the preparation procedure. We can express this in another way by saying (de Muynck, 1975) that the particles in AS states are *correlated* particles, the correlation being established by the preparation procedure.

By the equality (5) it is shown that

(6) $\text{AS} \rightarrow \text{OID}$.

So, if we insist on the possibility of OD, we will have to consider states that are *not* AS. Since, however, all experimental evidence *seems* to be consistent with AS, it might be questioned whether there is any point in pondering over observational distinguishability of identical particles. Perhaps we should content ourselves with the restriction of observations to the class of symmetrical observables.

As indicated before, we feel that such a restriction would not do justice to the possibility of giving a more detailed description of a system of identical particles in those situations where the preparation procedure prevents the establishment of the correlations described by

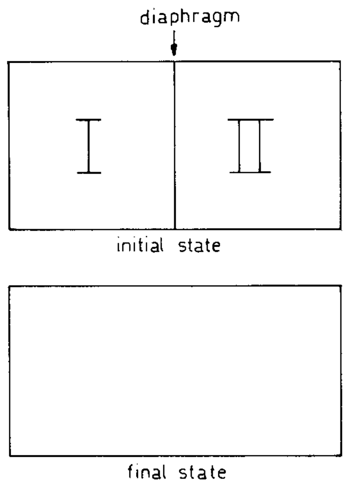

Fig. 2. The Gibbs problem.

AS. Such situations are obtained, for instance, if the system consists of two subsystems which are prepared by *independent* preparation procedures, as, in the case of the *Gibbs paradox*, in the initial state before the diaphragm has been removed (figure 2), or in case the electron guns of figure 1 have been placed in such positions that the electrons will not interact (figure 3). Admittedly it is possible to describe these situations also by means of (anti)symmetrized wave functions. This, however, has the consequence that the two physically distinct states of figure 2 (the

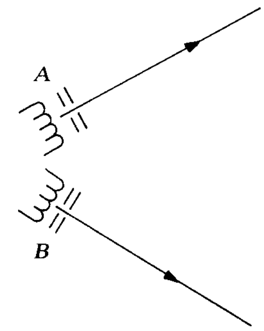

Fig. 3. Electron nonscattering experiment.

initial and final state) are described by the same wave function. This does not lead to inconsistencies as far as *symmetrical* observables are concerned, since the expectation values of symmetrical observables $f(x_1, \ldots, x_n)$ in the totally (anti)symmetrized state AS coincide with the expectation values of the product state $\mathrm{AS}^{\mathrm{I}} \otimes \mathrm{AS}^{\mathrm{II}}$, as long as the states of the subsystems I and II do not overlap. That is,

$$(7) \qquad \langle f(x_1, \ldots, x_n)\rangle_{\mathrm{AS}} = \langle f(x_1, \ldots, x_n)\rangle_{\mathrm{AS}^{\mathrm{I}} \otimes \mathrm{AS}^{\mathrm{II}}}$$

if

$$(8) \qquad f(x_1, \ldots, x_n) = f(P(x_1, \ldots, x_n))$$

is a symmetrical observable of the total system.

Notwithstanding the equivalence of both descriptions in this respect, it is felt that a wave function which is a product of the two wave functions of the subsystems would give a truer description of the initial state in both cases of figures 2 and 3. As a matter of fact, whereas in the AS state the expectation value $\langle x_i \rangle$ equals the expectation value of the center of mass of the *whole* system, in case of a product function it represents the position of the center of mass of the *sub*system to which the ith particle belongs. Evidently, the AS state does not yield the correct value of $\langle x_i \rangle$.

If the subsystems have not been in close contact with each other, nothing seems to impair the possibility of distinguishing the subsystems observationally by the positions of their centers of mass. This possibility even occurs in the *initial* state of the scattering experiment of figure 1, as long as the particles have not reached the scattering region. As is well known (e.g., Goldberger and Watson, 1964), it is not necessary to antisymmetrize the initial state in calculating scattering cross sections, if only the final state is AS. There even are situations where it is not at all necessary to antisymmetrize (Goldberger and Watson, 1964, p. 164). These situations occur if the exchange contribution to the scattering matrix is negligible, which will generally be the case if the particles remain far apart. As discussed above, this coincides with those situations which are unfavorable for the particles to get correlated. This justifies the use of product wave functions also from our point of view (de Muynck, 1975): in order to get correlated, the particles should be allowed to interact.

6. OID AND CAUSALITY

Whereas the arguments, given here, indicate the *possibility* of treating pre-interaction states as uncorrelated, and therefore as non-AS, without violating agreement with experimental data, by Mirman (1973) still another argument is supplied against total (anti)symmetrization of the wave function of the whole system as long as the wave functions of the subsystems do not yet overlap. Since AS expresses a correlation between the particles, the use of AS-functions would imply here a correlation between particles that never had a chance to interact. This, however, is felt as a contradiction. There exist two ways by means of which it is possible to lift this contradiction, (a) supposing the existence of an exchange interaction which establishes the correlation notwithstanding the absence of direct interaction, (b) the choice of a non-AS state as the pre-interaction state. Mirman chooses the second alternative, because the first alternative would violate causality: particle i would have a chance to be found in a region which it cannot reach, unless with a velocity exceeding the velocity of propagation of the interaction (which, supposedly, is the velocity of light).

Of course, the causality violation which is implied by AS can have observable consequences only if the particles are observationally distinguishable. As we have seen above, however, this is precisely the case if the systems have been created far apart! So we have to take Mirman's argument seriously, even if we are in a position to circumvent this causality problem by sticking to the "causal" formalism of second quantization. As discussed in section 4, this formalism gives a less detailed description of the system than the wave function formalism does. The equality (4), which has a clear meaning in this latter formalism, expresses a violation of causality if the particles, after their preparation, have not had enough time to become correlated in an AS state. If we are prepared to exploit the excess possibilities which are offered by the wave function formalism over second quantization, it is already the requirement of causality which forces us to describe the pre-interaction state as a product state of the wave functions of the individual particles. It should be noted that, although such product states have no counterparts in second quantization, this does not engender acausality for the second quantization formalism. This, however, is entirely due to the restriction of the class of observations which this formalism describes: symmetrical observables cannot distinguish a product state from the AS-state which is formed from it by means of (anti)symmetrization.

We would like to stress here that, in our opinion, the impossibility of obtaining a post-interaction state which is AS, if the state evolution is described by a Schrödinger equation with a *symmetric* Hamiltonian operator, is not a counterargument against the existence of a non-AS pre-interaction state. Contrary to Mirman (1973), who maintains that "the physical process in which distinguishability is destroyed, [...] should be understandable (and understood) within quantum mechanics", we believe that here a limitation is revealed which also the wave function formalism of quantum mechanics is subject to: quantum mechanics can *not* describe the process of correlation which takes place when the wave functions of the particles start to overlap. In the case of the Gibbs paradox (figure 2) this process of correlation does not seem different from the diffusion process that leads to the final equilibrium state of the gas. The problem of the reduction of such diffusion processes to reversible theories like classical or quantum mechanics, which is largely unsolved up to now, seems to be quite analogous to the problem of explaining the mechanism by which the wave function is antisymmetrized. Continuing this analogy, quantum mechanics, like thermodynamics, then, describes only states in which the particles have come to a certain equilibrium expressed by AS. A description of the correlation process leading, for SIP's, to AS, would have to await the conception of a theory intended to describe the microworld at a *sub*-quantum level. By Lyuboshitz and Podgoretskii (1969, 1971) it was shown that an analogous correlation mechanism for nonidentical particles does not influence the wave function.

7. KAPLAN'S DERIVATION OF AS FROM ID

In the remainder of this paper we will be interested in attempts to derive the converse of (6): OID → AS. One, by now trivial remark, is that this implication does not follow from the properties of the second quantization formalism itself. Although this formalism, as discussed above, corresponds to OID, it does not imply AS. As was demonstrated by Green (1953), an unindexed formalism can be conceived for other statistics than the statistics of bosons or fermions, viz., the so-called parastatistics. So the mere ignorance of particle indices does not commit us to AS.

It is claimed by Kaplan (1976) that a derivation of AS from ID is possible. We will give a short evaluation of this derivation in the light of the distinction made by us between OID and CID. Since Kaplan uses the indexed formalism, we may conclude that he presupposes CD. So, OID should be meant if he insists on the indistinguishability of identical particles. This restricts the physically relevant observables to the class of observables that are symmetric under permutation of the particle indices.

Kaplan's derivation is based on the assumption that one-particle operators like $f(x_i)$ represent directly observable properties. AS follows, if, analogous to (4), it is required that

$$(9) \qquad \langle f(x_1)\rangle = \cdots = \langle f(x_i)\rangle = \cdots = \langle f(x_n)\rangle.$$

Equalities (9) are conceived by Kaplan as "the mathematical formulation of the indistinguishability principle".

We do not challenge the derivability of AS from (9). In our opinion, however, it is the assumption (9) itself, which cannot be maintained. Either the particles are considered to be observationally indistinguishable, in which case the operators $f(x_i)$ have no physical relevance. Or the particles are considered to be observationally distinguishable. Then, nonsymmetrical operators like $f(x_i)$ may have physical relevance under certain circumstances. As discussed in section 5, however, these circumstances are obtained when the preparation of the system allows a description by means of a *non*-AS product state, which violates equality (9). Therefore, whenever the operators $f(x_i)$ have physical relevance, there is no a priori reason to require the equality of their expectation values. If the equality (9) seems to be a general feature of SIP's, this is merely so because the experimental situations to which quantum mechanics is applied, mostly correspond to AS-states describing correlated particles. This fact of experience, however, is not a necessary law, and actually has its exceptions. Therefore, the equality (9) should not be required as a necessary requirement for SIP's.

8. EQUIVALENT OBSERVABLES AND EQUIVALENT STATES

In order to be able to discuss a derivation of AS given by Sarry (1979), we will first write down some definitions which in our opinion give a good illustration of Sarry's underlying assumptions. Consider a Hilbert space (state space) $\mathcal{H}$, which can be part of a larger Hilbert space, and a set of observables $\mathcal{F}$ (self-adjoint operators on $\mathcal{H}$). For instance, in a two-particle system $\mathcal{H}$ can be $\mathcal{H}_1 \otimes \mathcal{H}_2$, $(\mathcal{H}_1 \otimes \mathcal{H}_2)_S$ or $(\mathcal{H}_1 \otimes \mathcal{H}_2)_A$, where $\mathcal{H}_i$ is a state space of an individual particle and S or A indicate (anti)symmetrization of the wave functions.

DEFINITION 1. ($A \overset{\mathcal{H}}{\sim} B$):
The observables A and B (elements of $\mathcal{F}$) are equivalent relative to $\mathcal{H}(A \overset{\mathcal{H}}{\sim} B)$ if

$$(10) \qquad \langle\Psi|A|\Psi\rangle = \langle\Psi|B|\Psi\rangle \quad \forall_{|\Psi\rangle\in\mathcal{H}}.$$

The equality (10) does not necessarily entail $A = B$: e.g., relative to $(\mathcal{H}_1 \otimes \mathcal{H}_2)_S$ we have $\langle x_1\rangle = \langle x_2\rangle$, so $x_1 \overset{(\mathcal{H}_1\otimes\mathcal{H}_2)_S}{\sim} x_2$, but $\langle x_1\rangle$ does not necessarily equal $\langle x_2\rangle$ relative to $\mathcal{H}_1 \otimes \mathcal{H}_2$, so $x_1 \neq x_2$.

DEFINITION 2. ($|\Phi\rangle \overset{\mathcal{F}}{\sim} |\Psi\rangle$):
The states $|\Phi\rangle$ and $|\Psi\rangle$ of $\mathcal{H}$ are equivalent relative to $\mathcal{F}(|\Phi\rangle \overset{\mathcal{F}}{\sim} |\Psi\rangle)$ if

$$(11) \qquad \langle\Psi|A|\Psi\rangle = \langle\Phi|A|\Phi\rangle \quad \forall_{A\in\mathcal{F}}.$$

An example: if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{F} = \{f(x_1, x_2) \mid f(x_1, x_2) = f(x_2, x_1),\ f \text{ real}\}$ then $\Psi(x_1, x_2)$ is equivalent to $\Psi(x_2, x_1)$ relative to $\mathcal{F}$.

DEFINITION 3. (separation):
The set of observables $\mathcal{F}$ is said to separate the states of $\mathcal{H}$ if $|\Phi\rangle \in \mathcal{H}$, $|\Psi\rangle \in \mathcal{H}$, $|\Phi\rangle \overset{\mathcal{F}}{\sim} |\Psi\rangle$ implies

$$(12) \qquad |\Phi\rangle = e^{i\alpha}|\Psi\rangle,\ \alpha \text{ a real constant.}$$

We can illustrate this definition again for a two-particle system, where e.g., the set of all (bounded) operators $A(1, 2)$ which are symmetrical under permutation of the indices, separates the states of $(\mathcal{H}_1 \otimes \mathcal{H}_2)_S$ and $(\mathcal{H}_1 \otimes \mathcal{H}_2)_A$ but does not separate the states of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

In order to connect these mathematical definitions with physics, we introduce the following definitions.

DEFINITION 4. (physically possible states):
A physically possible state of a system is a state which is not excluded by any observational evidence, as a description of the system.

We will call the (Hilbert) space which has the set of physically possible states as unit sphere, $\mathscr{H}_{\text{ph}}$.

DEFINITION 5. (physically relevant observables):
An operator $A \in \mathscr{F}$ corresponds to a physically relevant observable if a measuring procedure can be conceived to measure that observable.

The set of physically relevant observables will be indicated by $\mathscr{F}_{\text{ph}}$.

We will not be bothered by the inherent vagueness of definitions 4 and 5. Observational evidence is changing continually, as the class of physically relevant observables is growing and measurements of these observables are performed. As we saw before, which observables are physically relevant, can be a matter of dispute. This does not disturb us, however, because the present analysis is not based on which choice is made for the class of physically relevant observables. In the following our argumentation will derive from the assumption that it is necessary to remain faithful to the choice, once this has been made.

We will assume that projection operators of subspaces of $\mathscr{H}_{\text{ph}}$ are physically relevant observables. This seems reasonable because it allows us to check in which physically possible state the system is. Then, if

(13) $$\mathscr{G}_{\text{ph}} = \{A^{ij} \mid A^{ij} = \lambda \mid i\rangle\langle j \mid + \bar{\lambda} \mid j\rangle\langle i \mid\} \subset \mathscr{F}_{\text{ph}}$$

where $\{\mid i\rangle\}$ is a complete orthonormal set of vectors in $\mathscr{H}_{\text{ph}}$, and λ is a complex constant, $\mathscr{G}_{\text{ph}}$ is a set of physically relevant observables.

In case of a two-particle system, this set $\mathscr{G}_{\text{ph}}$ may for instance a priori have the following appearances:

if $\mathscr{H}_{\text{ph}} = \mathscr{H}_1 \otimes \mathscr{H}_2$: $\mathscr{G}_{\text{ph}} =$

$$\{A^{klmn} \mid A^{klmn} = \lambda|\Phi_k(x_1)\Phi_l(x_2)\rangle\langle\Phi_m(x_1)\Phi_n(x_2)| + \bar{\lambda}|\Phi_m(x_1)\Phi_n(x_2)\rangle\langle\Phi_k(x_1)\Phi_l(x_2)|\};$$

if $\mathscr{H}_{\text{ph}} = (\mathscr{H}_1 \otimes \mathscr{H}_2)_S$: $\mathscr{G}_{\text{ph}} =$

$$\{A_S^{klmn} \mid A_S^{klmn} = \lambda|\Phi_{kl}^{(S)}(x_1, x_2)\rangle\langle\Phi_{mn}^{(S)}(x_1, x_2)| + \bar{\lambda}|\Phi_{mn}^{(S)}(x_1, x_2)\rangle\langle\Phi_{kl}^{(S)}(x_1, x_2)|\}$$

where $\{\Phi_{kl}^{(S)}(x_1, x_2)\}$ is a complete orthonormal set of symmetrical wave functions.

Using definitions 2 and 3 it can be demonstrated easily that such a set $\mathscr{G}_{\text{ph}}$ separates $\mathscr{H}_{\text{ph}}$.

9. DISCUSSION OF SARRY'S DERIVATION OF AS

We now examine Sarry's definition (1979) of a system of identical particles.

DEFINITION 6. SIP (Sarry):
If the mean value of the operator A of any observable quantity of a system of particles remains unchanged for arbitrary permutations of the indices of the quantities x_i in the function $|\Psi\rangle$ over which this mean value is calculated, then the particles of this system should be regarded as identical (indistinguishable).

It is consistent with Sarry's following derivation if we interpret this according to

$$(14)\qquad \forall_{A\in\mathcal{F}_{\mathrm{ph}}}\langle\Phi|A|\Phi\rangle=\langle\Psi|A|\Psi\rangle,\quad |\Phi\rangle=P|\Psi\rangle,\ |\Phi\rangle,|\Psi\rangle\in\mathcal{H}_{\mathrm{ph}}.$$

Taking into account the unitarity, $P^{\dagger}=P^{-1}$, of the permutation operators, and using definition 1 we can conclude that Sarry's definition of an SIP amounts to

$$(15)\qquad \forall_{A\in\mathcal{F}_{\mathrm{ph}}}P^{-1}AP\overset{\mathcal{H}_{\mathrm{ph}}}{\sim}A.$$

From this condition we would like to extract information about P. But first, if we take a look at definition 1 again, we can show that, by virtue of the superposition principle and the arbitrariness of $|\Psi\rangle$ in the definition, the equality (10) for $\mathcal{H}=\mathcal{H}_{\mathrm{ph}}$ is equivalent to

$$\langle m|A|n\rangle=\langle m|B|n\rangle\quad\forall_{|m\rangle,|n\rangle\in\mathcal{H}_{\mathrm{ph}}}.$$

Therefore, Sarry's definition (14) of an SIP, which is the same as

$$\forall_{A\in\mathcal{F}_{\mathrm{ph}}}\langle\Phi|AP|\Psi\rangle=\langle\Phi|PA|\Psi\rangle$$

(assuming that both $|\Psi\rangle$ and $P|\Psi\rangle$ are elements of $\mathcal{H}_{\mathrm{ph}}$), can be interpreted according to definition 1 as

$$(16)\qquad \forall_{A\in\mathcal{F}_{\mathrm{ph}}}AP\overset{\mathcal{H}_{\mathrm{ph}}}{\sim}PA.$$

Using matrix-representations of P and A, this can be written as

$$\forall_{A\in\mathcal{F}_{\mathrm{ph}}}\sum_j A_{mj}P_{jn}=\sum_j P_{mj}A_{jn}$$

where

> $A_{mj} = \langle m | A | j \rangle$, $P_{mj} = \langle m | P | j \rangle$, $| m \rangle$, $| j \rangle$ elements of a complete orthonormal set of vectors in $\mathscr{H}_{\text{ph}}$.

Substituting the set $\mathscr{G}_{\text{ph}}$ which was introduced in (13), we find that

(17) $$P_{ij} = e^{i\alpha}\delta_{ij} \quad \text{or} \quad P \overset{\mathscr{H}_{\text{ph}}}{\sim} e^{i\alpha} I.$$

At this point Sarry concludes that, because of (17), $\mathscr{H}_{\text{ph}}$ contains either fully symmetrical or fully antisymmetrical wave functions, that is $\mathscr{H}_{\text{ph}}$ equals $\mathscr{H}_S$ or $\mathscr{H}_A$. This is so, because (17) implies one-dimensionality of the representations on $\mathscr{H}_{\text{ph}}$ of the permutation group, which commits us to either the symmetrical or the antisymmetrical representation of this group.

Our critique of the present derivation of AS rather follows the same line as our objection against Kaplan's derivation (section 7). Again, we do not challenge the derivation as it stands, but we will demonstrate that also here an unwarranted assumption is made, analogous to the equality (9). To this end we direct our attention to the Hilbert space of physical state functions $\mathscr{H}_{\text{ph}}$. If we take $\mathscr{H}_{\text{ph}}$ to be equal to $\mathscr{H}_A$ or $\mathscr{H}_S$, then, with $\mathscr{G}_{\text{ph}}$ defined by (13), we have

(18) $$P^{-1} A^{ij} P = A^{ij} \quad \forall_{A^{ij} \in \mathscr{G}_{\text{ph}}}.$$

The equality (18) is trivially fulfilled since for this choice of $\mathscr{H}_{\text{ph}}$, because of the symmetry properties of the states $| i \rangle$ and $| j \rangle$, the operators A^{ij} of $\mathscr{G}_{\text{ph}}$ are all symmetrical under a permutation of the particles. Also (17) is trivially fulfilled.

Making this choice for $\mathscr{H}_{\text{ph}}$, however, would clearly be question-begging since, then, we are *assuming* what has to be proven. The question is, then, whether in Sarry's derivation of AS the space $\mathscr{H}_{\text{ph}}$ is restricted to $\mathscr{H}_S$ or $\mathscr{H}_A$ in a legitimate way, without this restriction being assumed beforehand.

If this restriction of $\mathscr{H}_{\text{ph}}$ is dropped, the operators A^{ij} of $\mathscr{G}_{\text{ph}}$ are no longer limited to the symmetrical ones obeying (18), and the derivation of (17) can now be performed in the way outlined by Sarry. However, if nonsymmetrical observables are thought to be physically relevant, the situation is quite analogous to the one we met in section 7: we have no reason to require the physical equivalence (15) if A is such a nonsymmetrical observable. Since (17) is a consequence of (15), this blocks the derivation of AS.

We close this section by noting that, although

(19) *not*: $P^{-1}A^{ij}P \overset{\otimes_i \mathscr{H}_i}{\sim} A^{ij}$, if $\bigotimes_i \mathscr{H}_i$ is the tensor product of all single particle Hilbert spaces of the particles contained in the SIP, and A^{ij} are the operators defined by (13) with respect to $\mathscr{H}_{\text{ph}} = \bigotimes_i \mathscr{H}_i$ (since this would be tantamount to (18)),

for these nonsymmetrical operators the equivalences

(20) $P^{-1}A^{ij}P \overset{\mathscr{H}_S}{\sim} A^{ij}$, $P^{-1}A^{ij}P \overset{\mathscr{H}_A}{\sim} A^{ij}$

obtain. The relations (20) are to be compared to (5). They express the indistinguishability of the identical particles if the state is an AS-state, that is, the implication (6). A proof of the converse of this implication, however, requires the stronger form of equivalence as negated in (19).

10. CONCLUSIONS

In the quantum mechanical literature we can find many different views with respect to the relation between the (anti)symmetrization postulate and the alleged indistinguishability of identical particles, ranging from complete equivalence to complete independence of the two notions. Such a situation calls for a closer analysis of the concepts which are involved. In doing so we are lead to the introduction of two different notions of distinguishability, viz., conceptual and observational distinguishability. We come to the conclusion that only the second kind is involved when identical particles are thought *not* to be distinguishable, the first kind being presupposed both in classical and quantum mechanics.

Observational (in)distinguishability has to do with the class of observables that can be measured on the system of identical particles. Observables that are invariant under a permutation of the particle variables cannot distinguish between the particles. For this reason the second quantization formalism can only describe particles that are thought to be indistinguishable. However, the wave function formalism of quantum mechanics allows also observables that are not invariant under permutations. Although in an (anti)symmetrical state the particles are indistinguishable also with respect to such observables, there are physical situations in which a system of identical particles should be described by a state that is *not* (anti)symmetrical. In such states nonsymmetrical observables can be used to distinguish the particles.

Although it turns out that (anti)symmetry of the wave functions entails observational indistinguishability, the converse is not true. Two attempts to derive (anti)symmetry from indistinguishability are analyzed and shown to be defective: either we may consider only symmetrical observables as physically relevant, in which case a permutation does not entail any requirement to be obeyed by the wave function; or, also nonsymmetrical observables are taken into account, in which case, however, we accept the possibility of observational distinguishability of the particles. These two positions should not be mixed up.

The generality of our arguments against derivability of AS from ID seems to warrant the claim that AS and ID have quite different physical origins. Whereas ID refers to the possibilities of *observation*, AS is related to the *preparation* of the system of identical particles. It is tempting to view such a preparation process as analogous to the diffusion process by which two volumes of a gas are mixed (de Muynck, 1975). An interpretation of the quantum mechanical state as an equilibrium state of some diffusion process, which forces itself on us as an explanation of (anti)symmetry of the wave function, might be instrumental in explaining also other outstanding problems that pose itself in the foundation of quantum mechanics, as for instance the measurement problem and the Einstein-Podolsky-Rosen paradox.

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